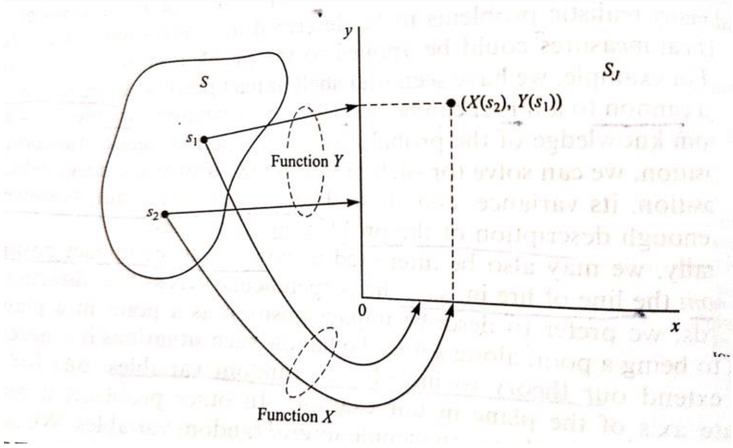
UNIT-II
MULTIPLE RANDOM VARIABLES

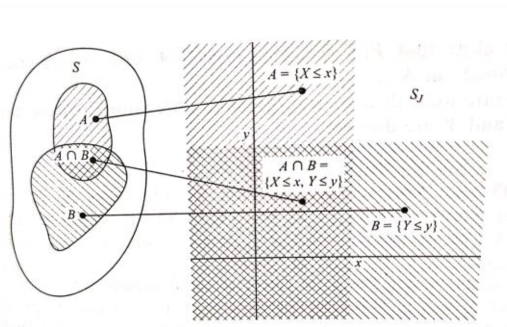
Multiple Random Variables: Vector random variables, Joint distribution function and properties, Marginal distribution functions, Joint density function and properties, Marginal density functions, Joint Conditional distribution and density functions, statistical independence, Distribution and density of sum of random variables, Central limit theorem.

Operations on Multiple Random Variables: Expected value of a function of random variables, Joint moments about the origin, Correlation, Joint central moments, Covariance, Correlation coefficient, Joint characteristic function and properties, Jointly Gaussian random variables-two and N random variables, properties

VECTOR RANDOM VARIABLS

- ✓ suppose two random variables X and Y are defined on a sample space S, where specific values of X and Y are denoted by x and y respectively.
- ✓ Then any ordered pair of numbers (x,y) may be conveniently considered to be a random point in the xy-plane.





Joint Probability distribution function

Consider two random variables X and Y with elements $\{x\}$ and $\{y\}$ in xy plane.

Let two events $A = \{X \leq x\}$ and $B = \{Y \leq y\}$ then the Joint Probability distribution function It gives information about probability of event $\{X \leq x\}$

$$F_{XY}(x, y) = \text{P}\{X \leq x, Y \leq y\} = P(A \cap B)$$

Properties of Joint Probability distribution function

$i)$ $F_{XY}(-\infty, -\infty) = 0$

$ii)$ $F_{XY}(x, -\infty) = 0$

$iii)$ $F_{XY}(-\infty, y) = 0$

Proof:

It is known that

$$i)\ F_{XY}(x, y) = \text{P}\{X \leq x, Y \leq y\}$$

$$F_{XY}(-\infty, -\infty) = \text{P}\{X \leq -\infty, Y \leq -\infty\}$$

$$= \text{P}\{X \leq -\infty\ \cap Y \leq -\infty\}$$

$$F_{XY}(-\infty, -\infty) = 0$$

$$ii)\ F_{XY}(x, y) = \text{P}\{X \leq x, Y \leq y\}$$

$$F_{XY}(x, -\infty) = \text{P}\{X \leq x, Y \leq -\infty\}$$

$$= \text{P}\{X \leq x\ \cap Y \leq -\infty\}$$

$$F_{XY}(x, -\infty) = 0$$

$$iii)\quad F_{XY}(x, y) = \text{P}\{X \leq x, Y \leq y\}$$

$$F_{XY}(-\infty, y) = \text{P}\{X \leq -\infty, Y \leq y\}$$

$$= \text{P}\{X \leq -\infty\ \cap Y \leq y\}$$

$$F_{XY}(-\infty, y) = 0$$

2. $F_{XY}(\infty, \infty) = 1$

Proof:

It is known that

$$F_{XY}(x, y) = \text{P}\{X \leq x, Y \leq y\}$$

$$F_{XY}(\infty, \infty) = \text{P}\{X \leq \infty, Y \leq \infty\}$$

$$= \text{P}\{X \leq \infty \cap Y \leq \infty\}$$

$$= \text{P}\{S \cap S\} = \text{P}\{S\} = 1$$

$$F_{XY}(\infty, \infty) = 1$$

3. The joint Probability distribution function is always define between 0 and 1.

i.e; $0 \leq F_{XY}(x, y) \leq 1$

4. Marginal distribution functions

$$F_X(x) = F_{XY}(x, \infty)$$

$$F_Y(y) = F_{XY}(\infty, y)$$

Proof:

It is known that

$F_{XY}(x, y) = \text{P}\{X \leq x, Y \leq y\}$

$F_{XY}(x, \infty) = \text{P}\{X \leq x, Y \leq \infty\}$

$= \text{P}\{X \leq x \cap Y \leq \infty\}$

$= \text{P}\{X \leq x \cap s\}$

$= \text{P}\{X \leq x\}$

$F_{XY}(x, \infty) = F_X(x)$

$F_{XY}(x, y) = \text{P}\{X \leq x, Y \leq y\}$

$F_{XY}(\infty, y) = \text{P}\{X \leq \infty, Y \leq y\}$

$= \text{P}\{s \cap Y \leq y\}$

$= \text{P}\{Y \leq y\}$

$F_{XY}(\infty, y) = F_Y(y)$

Joint Probability Density Function:

It gives information about the joint occurrence of events at given values of X and Y

- ✓ The joint probability density function is the partial derivatives of joint distribution.

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x\, \partial y}$$

when X and Y are discrete random variables, joint density function is

$$f_{XY}(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} P\left(X = x_i, Y = y_j\right) \delta\left(X - x_i, Y - y_j\right)$$

Properties of Joint Probability density function

1. Joint probability density function is a non-negative quantity

$$f_{XY}(x,y) \geq o$$

Proof

From the definition

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x\, \partial y}$$

As the distribution function is anon non-decreasing function slope is always positive. Hence the joint probability density function is a non-negative quantity.

2. The area under the probability density function is unity.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y) dx\, dy = 1$$

Proof:

We know that

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x\, \partial y}$$

Integrate on both sides w.r.to x

$$\int_{-\infty}^{\infty} f_{XY}(x,y)dx = \int_{-\infty}^{\infty} \frac{\partial^2 F_{XY}(x,y)}{\partial x\, \partial y} dx$$

$$= \frac{\partial}{\partial y}\int_{-\infty}^{\infty} \frac{\partial}{\partial x} F_{XY}(x,y)dx$$

$$= \frac{\partial}{\partial y}\left[F_{X,Y}(x,y)\right]_{-\infty}^{\infty}$$

$$= \frac{\partial}{\partial y}(F_{XY}(\infty,y) - F_{XY}(-\infty,y))$$

Integrate on both sides w.r.to y

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dx\, dy = \int_{-\infty}^{\infty} \frac{\partial}{\partial y} F_Y(y)dy$$

$$= [F_Y(y)]_{-\infty}^{\infty}$$

$$= F_Y(\infty) - F_Y(-\infty) = 1$$

$$= \frac{\partial}{\partial y} F_Y(y)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dx\, dy = 1$$

3. The joint probability distribution function can be obtained from the knowledge of joint density function.

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x,y)dx\, dy$$

Proof

$$we\ know\ \ f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x\, \partial y}$$

Integrating on both sides w. r. to x

$$\int_{-\infty}^{x} f_{XY}(x,y)dx = \int_{-\infty}^{x} \frac{\partial^2 F_{XY}(x,y)}{\partial x\, \partial y} dx$$

$$\frac{\partial}{\partial y}\int_{-\infty}^{x} \frac{\partial}{\partial x} F_{XY}(x,y)dx$$

$$= \frac{\partial}{\partial y}\left[F_{X,Y}(x,y)\right]_{-\infty}^{x}$$

$$= \frac{\partial}{\partial y}\left(F_{XY}(x,y) - F_{XY}(-\infty,y)\right)$$

$$= \frac{\partial}{\partial y} F_{XY}(x,y)$$

Integrate on both sides w.r.to y

$$\int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(x,y)\, dx\, dy = \int_{-\infty}^{y} \frac{\partial}{\partial y} F_{XY}(x,y) dy$$

$$= [F_{XY}(x,y)]_{-\infty}^{y}$$

$$= F_{XY}(x,y) - F_{XY}(x,-\infty)$$

$$= F_{XY}(x,y)$$

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x,y) dx\, dy$$

4. The probability of event $\{x_1 < X \le x_2,\ y_1 < Y \le y_2\}$ can be obtained from the knowledge of joint density function.

$$P\{x_1 < X \le x_2,\ y_1 < Y \le y_2\} = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{XY}(x,y) dx\, dy$$

Proof:

Consider

$$= \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{XY}(x,y) dx\, dy$$

$$= \int_{x_1}^{x_2}\int_{y_1}^{y_2} \frac{\partial^2 F_{XY}(x,y)}{\partial x\, \partial y} dx\, dy$$

Changing the order of integration

$$\int_{y_1}^{y_2} \frac{\partial}{\partial y} \int_{x_1}^{x_2} \frac{\partial}{\partial x} F_{XY}(x,y) dx\, dy$$

$$\int_{y_1}^{y_2} \frac{\partial}{\partial y} \left[F_{X,Y}(x,y)\right]_{x_1}^{x_2} dy$$

$$\int_{y_1}^{y_2} \frac{\partial}{\partial y} (F_{XY}(x_2,y) - F_{XY}(x_1,y))dy$$

$$= [(F_{XY}(x_2,y) - F_{XY}(x_1,y))]_{y_1}^{y_2}$$

$$= F_{XY}(x_2,y_2) - F_{XY}(x_1,y_2) - F_{XY}(x_1,y_2) + F_{XY}(x_1,y_1)$$

Marginal density functions

$$\boldsymbol{f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy}$$

$$\boldsymbol{f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx}$$

$$we\ know\ \ f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x\, \partial y}$$

Integrate on both sides w.r.to x

$$\int_{-\infty}^{\infty} f_{XY}(x,y)dx = \int_{-\infty}^{\infty} \frac{\partial^2 F_{XY}(x,y)}{\partial x\, \partial y} dx$$

$$= \frac{\partial}{\partial y} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} F_{XY}(x,y)dx$$

$$= \frac{\partial}{\partial y} \left[F_{X,Y}(x,y)\right]_{-\infty}^{\infty}$$

$$= \frac{\partial}{\partial y} (F_{XY}(\infty,y) - F_{XY}(-\infty,y))$$

$$= \frac{\partial}{\partial y} F_Y(y)$$

$$= f_Y(y)$$

$$\boldsymbol{f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx}$$

Integrate on both sides w.r.to y

$$\int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_{-\infty}^{\infty} \frac{\partial^2 F_{XY}(x,y)}{\partial x\, \partial y} dy$$

$$= \frac{\partial}{\partial x}\int_{-\infty}^{\infty} \frac{\partial}{\partial y} F_{XY}(x,y)dy$$

$$= \frac{\partial}{\partial x}\left[F_{X,Y}(x,y)\right]_{-\infty}^{\infty}$$

$$= \frac{\partial}{\partial x}(F_{XY}(x,\infty) - F_{XY}(x,-\infty))$$

$$= \frac{\partial}{\partial x} F_X(x)$$

$$= f_X(y)$$

$$\boldsymbol{f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy}$$

CONDITIONAL PROBABILITY

It is the probability of an event 'A' based on the occurrence of another event 'B'.

Let A and B are two events then the conditional probability of A upon the occurrence of B is given as

$$P\left({}^{A}\!/_{B}\right) = \frac{P(A \cap B)}{P(B)}$$

Similarly, the conditional probability of B upon the occurrence of A is given as

$$P\left({}^{B}\!/_{A}\right) = \frac{P(A\cap B)}{P(A)}$$

CONDITIONAL DISTRIBUTION FUNCTION

The concept of conditional probability is extended for random variable also.

Let 'X' be a random variable then the conditional distribution function is defined as

$$F_X\left({}^{x}\!/_{B}\right) = \mathrm{P}\{X \leq x \cap B\}$$

Properties of Probability Distribution Function

1. $F_X\left({}^{-\infty}/_B\right) = 0$
2. $F_X\left({}^{\infty}/_B\right) = 1$
3. $0 \leq F_X\left({}^{x}/_B\right) \leq 1$
4. $F_X(x_2/B) \geq F_X(x_1)/B \; when \; x_2 > x_1$
5. $P\left\{{}^{(x_1 < X \leq x_2)}/_B\right\} = F_X(x_2/B) - F_X(x_1/B)$

CONDITIONAL DENSITY FUNCTION

The derivative of conditional distribution function is called conditional density function.

It gives the conditional probability of an event at a specific value.

$$f_X\left({}^{x}/_B\right) = \frac{dF_X\left({}^{x}/_B\right)}{dx}$$

Properties

1. $f_X\left({}^{x}/_B\right) \geq o$
2. $\int_{-\infty}^{\infty} f_X\left({}^{x}/_B\right) dx = 1$
3. $F_X(x/B) = \int_{-\infty}^{x} f_X\left({}^{x}/_B\right) dx$
4. $P\{x_1 < X \leq x_2\} = \int_{x_1}^{x_2} f_X\left({}^{x}/_B\right) dx$

JOINT CONDITIONAL DISTRIBUTION AND DENSITY FUNCTIONS

Two conditions are defined called as

1. Point conditioning
2. Interval conditioning

Based on the values taken by random variable ‘X’

if ‘Y’ takes a single value then it is called Point conditioning

if ‘Y’ takes a range of values then it is called Interval conditioning

<u>Point conditioning:</u>

Under this the random variable ‘Y’ takes a single value such that

$$y - \Delta y \ \leq Y \ \leq y + \Delta y$$

$$P\left({}^{A}/_{B}\right) = \frac{P(A \cap B)}{P(B)}$$

$F_X\left({}^{x}/_{B}\right) = \text{ P}\{X \leq x \cap B\}$

$$F_X\left({}^{x}/_{y - \Delta y \ \leq Y \ \leq y + \Delta y}\right) = \frac{\text{P}\{(X \leq x) \cap (y - \Delta y \ \leq Y \ \leq y + \Delta y)\}}{\text{P}(y - \Delta y \ \leq Y \ \leq y + \Delta y)}$$

It is known that the distribution function is the integral of density function

$$F_X\left({}^{x}/_{y - \Delta y \ \leq Y \ \leq y + \Delta y}\right) = \frac{\int_{y-\Delta y}^{y+\Delta y} \int_{-\infty}^{x} f_{XY}(x, y) dx \, dy}{\int_{y-\Delta y}^{y+\Delta y} f_Y(y) \, dy}$$

$$\Delta y \ \longrightarrow 0 \ \ F_X\left({}^{x}/_{y - \Delta y \ \leq Y \ \leq y + \Delta y}\right) = \frac{\int_{-\infty}^{x} f_{XY}(x, y) dx \ \ \int_{y-\Delta y}^{y+\Delta y} \ dy}{f_Y(y) \ \int_{y-\Delta y}^{y+\Delta y} \ dy}$$

$$= \frac{\int_{-\infty}^{x} f_{XY}(x, y) dx \ 2\Delta y}{f_Y(y) \ \ 2\Delta y} = \frac{\int_{-\infty}^{x} f_{XY}(x, y) dx}{f_Y(y)}$$

Conditional density function is

$$f_X\left({}^{x}/_{Y}\right) = \frac{d \ F_X\left({}^{x}/_{Y}\right)}{dx} = \frac{\frac{d}{dx}\int_{-\infty}^{x} f_{XY}(x, y) dx}{f_Y(y)}$$

$$f_X\left({}^{x}/_{Y}\right) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

Similarly

$$f_Y\left({}^{y}/_{X}\right) = \frac{f_{XY}(x, y)}{f_X(x)}$$

<u>Interval conditioning:</u>

Under this the random variable ‘Y’ takes the range of values such that

$$y_a \leq Y \ \leq y_b$$

$$F_X\left({}^{x}/_{y_a} \le Y \le y_b\right) = \frac{\int_{y_a}^{y_b}\int_{-\infty}^{x} f_{XY}(x,y)dx\,dy}{\int_{y_a}^{y_b} f_Y(y)\,dy}$$

$$F_X\left({}^{x}/_{y_a} \le Y \le y_b\right) = \frac{\int_{y_a}^{y_b} f_{XY}(x,y)\,dy}{\int_{y_a}^{y_b} f_Y(y)\,dy}$$

$$F_X\left({}^{x}/_{y_a} \le Y \le y_b\right) = \frac{\int_{y_a}^{y_b} f_{XY}(x,y)\,dy}{\int_{y_a}^{y_b}\int_{-\infty}^{\infty} f_{XY}(x,y)dx\,dy}$$

STATISTICAL INDEPENDENCE OF RANDOM VARIABLES:

Consider two random variables X and Y by defining the events $A = \{X \le x\}$ and $B = \{Y \le y\}$ for two real numbers x *and* y

Two random variables are said to be statistical independent then

$$\text{P}\{X \le x, Y \le y\} = \text{P}\{X \le x\}\ \ \text{P}\{Y \le y\}$$

From distribution function

$$F_{XY}(x,y) = F_X(x)\,F_Y(y)$$

From density function

$$f_{XY}(x,y) = f_X(x)\,f_Y(y)$$

From conditional density function

$$f_X({}^{x}/_{y}) = \frac{f_{XY}(x,y)}{f_Y(y)} == \frac{f_X(x)\,f_Y(y)}{f_Y(y)} = f_X(x)$$

$$f_X({}^{y}/_{x}) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{f_X(x)\,f_Y(y)}{f_X(x)} = f_Y(y)$$

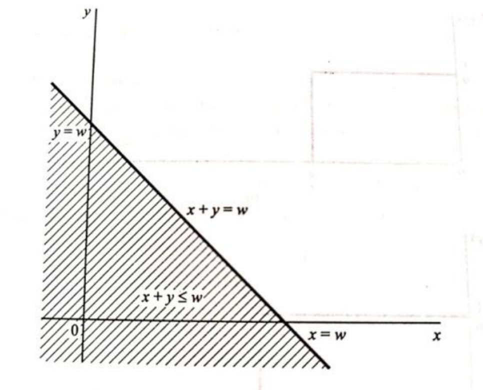
DISTRIBUTION AND DENSITY OF A SUM OF RANDOM VARIABLES

In real time applications, the received signal is a sum of designed original signal and noise. In such a case the information about the probability of combine signal will help in analysing the communication system.

Let 'W' be a random variable equal to sum of two independent random variables X and Y

$$W = X + Y$$

The density function of sum of two random variables is the area under the given curve $W = X + Y$



$$F_W(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f_{XY}(x, y) dx\, dy$$

When X and Y are statically independent

$$F_W(w) = \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{w-y} f_X(x)\, dx\, dy$$

Differentiating (using Leibniz's rule) w.r.to 'w' on both side

$$\frac{d}{dw} F_W(w) = \frac{d}{dw} \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{w-y} f_X(x)\, dx\, dy$$

$$f_W(w) = \int_{-\infty}^{\infty} f_Y(y) \frac{d}{dw} \int_{-\infty}^{w-y} f_X(x)\, dx\, dy$$

$$\frac{\partial}{\partial z} \int_{a(z)}^{b(z)} f(x, z)\, dx = \int_{a(z)}^{b(z)} \frac{\partial f}{\partial z}\, dx + f(b(z), z) \frac{\partial b}{\partial z} - f(a(z), z) \frac{\partial a}{\partial z}.$$

$$f_W(w) = \int_{-\infty}^{\infty} f_Y(y) [f_X(w - y) - f_X(-\infty)]\, dy$$

$$f_W(w) = \int_{-\infty}^{\infty} f_Y(y)\, f_X(w - y)\, dy$$

The above expression is recognized as a convolution integral.

$$f_W(w) = f_X(x) \otimes f_Y(y)$$

The density function of the sum of two statistically independent random variables is the convolution of their individual density functions.

If there are ‘n’ number of statistically independent random variables then

$$f_{X_1+X_2+\cdots X_n}(x) = f_{X_1}(x_1) * f_{X_2}(x_2) * f_{X_3}(x_3) \dots \dots * f_{X_n}(x_n)$$

OPERATIONS ON MULTIPLE RANDOM VARIABLES:

Expected Value or Mean

When more than a single random variable is involved, expectation must be taken with respect to all the variables involved.

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\, dx\, dy$$

JOINT MOMENTS:

Moments are the measure of deviation of a random variable from a reference value.

Joint moments indicate the deviation of a multiple random variables from a reference value.

Joint moments about the origin

The expected value of a function of a form $g(x,y) = X^n Y^k$ is called joint moment about the origin.

$$m_{nk} = E[X^n Y^k]$$

$$m_{nk} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{XY}(x,y)\, dx\, dy$$

The order of joint moment is the sum of individual orders n and k.

i.e: order=n+k

First order joint moments:

$$m_{nk} = E[X^n Y^k]$$

$$m_{10} = E[X^1 Y^0] = E(X) = m_1 = \int_{-\infty}^{\infty} x.f_X(x)\ dx$$

$$m_{01} = E[X^0 Y^1] = E(Y)$$

Note: The first order joint moments are equal respective individual expected value.

Second order joint moments:

$$m_{02} = E[\,Y^2]$$

$$m_{20} = E[\,X^2]$$

The second order joint moment m_{11} is called as Correlation.

$$R_{XY} = m_{11} = E[X\,Y]$$

$$m_{11} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,y\,f_{XY}(x,y)\,dx\,dy$$

Correlation is a measure of similarity between two (or) more random variables.

If two random variables are said to be statistical independent then

$$R_{XY} = E[X\,Y] = E[X]\;\,E[Y]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,y\,f_{XY}(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,y\,f_X(x)\,f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} x\,f_X(x)\,dx \int_{-\infty}^{\infty} y\,f_Y(y)\;\,dy = E[X]\;\,E[Y]$$

Note: when two random variables are orthogonal $R_{XY} = 0$

Joint central moments

The expected value of a given function $g(x) = (X - \bar{X})^n (Y - \bar{Y})^K$ is called joint central moment of two random variables.

$$\mu_{nk} = E[(X - \bar{X})^n\,(Y - \bar{Y})^k]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \bar{X})^n\;\,(y - \bar{Y})^k\,f_{XY}(x,y)\,dx\,dy$$

Properties of central moments:

1. The zero order joint central moment is ‘1’.
2. The first order joint central moment is ‘zero’.

$$\mu_{01} = \mu_{10} = 1$$

3. The second order joint central moment

$$\mu_{nk} = E[(X - \bar{X})^n \, (Y - \bar{Y})^k]$$

$$\mu_{20} = E[(X - \bar{X})^2] = {\sigma_X}^2$$

$$\mu_{02} = E[\, (Y - \bar{Y})^2] = {\sigma_Y}^2$$

4. The second order joint central moment is called as covariance between two random variables X and Y.

$$\mu_{11} = E[(X - \bar{X})(Y - \bar{Y})] = C_{XY}$$

Covariance is a measure of change in random variable with another one. It indicates how to random variables vary together.

Properties of Covariance:

1. If X and Y are two random variables then the Covariance between them is given as

$$C_{XY} = R_{XY} - E[X]\ \ E[Y]$$

2. If X and Y are two statistical independent random variables then

$$C_{XY} = 0$$

$$C_{XY} = R_{XY} - E[X]\ \ E[Y]$$

If two random variables are said to be statistical independent then

$$R_{XY} = E[X\ Y] = E[X]\ \ E[Y]$$

$$C_{XY} = E[X]\ \ E[Y] - E[X]\ \ E[Y] = 0$$

3. Let X and Y be two random variables then

$$var[X + Y] = var[X] + var[Y] + 2C_{XY}$$

$$var[X - Y] = var[X] + var[Y] - 2C_{XY}$$

Correlation coefficient:

It is defined as

$$\rho = \frac{\mu_{11}\ (or)\ C_{XY}}{\sqrt{\mu_{20}\quad \mu_{02}}}$$

$$\rho = \frac{\mu_{11}}{\sigma_X\ \sigma_Y}$$

JOINT CHARACTERISTICS FUNCTION

The expected value of the joint function $g(x,y) = e^{j\omega_1 X} e^{j\omega_2 Y}$ is called joint characteristics function.

$$\phi_{XY}(\omega_{1,}\omega_2) = E\left[e^{j\omega_1 X} e^{j\omega_2 Y}\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\omega_1 X}\ e^{j\omega_2 Y} f_{XY}(x,y)\ \mathrm{dx\ dy}$$

$$f_{XY}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega_1 X}\ e^{-j\omega_2 Y}\ \phi_{XY}(\omega_{1,}\omega_2)\ \mathrm{d}\omega_1 \mathrm{d}\omega_2$$

Joint characteristics function and joint density function are Fourier transform pairs with the sign of the variable are reversed.

Properties of joint characteristics function:

1. The marginal characteristics function can be obtained from the knowledge of joint characteristics function

$$\phi_X(\omega_1) = \phi_{XY}(\omega_{1,}0)$$

$$\phi_Y(\omega_2) = \phi_{XY}(0,\omega_2)$$

proof:

We know that

$$\phi_{XY}(\omega_{1,}\omega_2) = E\left[e^{j\omega_1 X} e^{j\omega_2 Y}\right]$$

Let $\omega_2 = 0$

$$\phi_{XY}(\omega_{1,}0) = E\left[e^{j\omega_1 X}\right] = \phi_X(\omega_1)$$

Let $\omega_1 = 0$

$$\phi_{XY}(0,\omega_2) = E\left[e^{j\omega_2 Y}\right] = \phi_Y(\omega_2)$$

2. If X and Y are two statistical independent random variables then their joint characteristics function is the product of individual characteristics functions

$$\phi_{XY}(\omega_{1,}\omega_2) = \phi_X(\omega_1)\ \phi_Y(\omega_2)$$

$proof$:

We know that

$$\phi_{XY}(\omega_{1,}\omega_2) = E[e^{j\omega_1 X} e^{j\omega_2 Y}]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\omega_1 X}\ e^{j\omega_2 Y} f_{XY}(x,y)\ \mathrm{dx\ dy}$$

If X and Y are two statistical independent random variables then

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\omega_1 X}\ e^{j\omega_2 Y} f_{XY}(x,y)\ \mathrm{dx\ dy}$$

$$= \int_{-\infty}^{\infty} e^{j\omega_1 X}\ f_X(x)\ \mathrm{dx} \int_{-\infty}^{\infty} e^{j\omega_2 Y}\ f_Y(y)\ \mathrm{dy}$$

$$\phi_{XY}(\omega_{1,}\omega_2) = \phi_X(\omega_1)\ \phi_Y(\omega_2)$$

3. If X and Y are two statistical independent random variables then the joint characteristics function of sum of random variables is the product of individual characteristics functions

$$\phi_{X+Y}(\omega) = \phi_X(\omega)\ \phi_Y(\omega)$$

$proof$:

$$\phi_{X+Y}(\omega) = E[e^{j\omega(X+Y)}]$$
$$= E[e^{j\omega X}\ \ e^{j\omega Y}]$$

If X and Y are two statistical independent random variables then

$$\phi_{X+Y}(\omega) = E[e^{j\omega X}]\ E[e^{j\omega Y}]$$
$$\phi_{X+Y}(\omega) = \phi_X(\omega)\ \phi_Y(\omega)$$

4. The joint moments of multiple random variable can be obtained from the knowledge of joint characteristic function is

$$m_n = (-j)^{n+k} \frac{\partial^{n+k}}{\partial \omega_1{}^n \, \partial \omega_2{}^k} \phi_{XY}(\omega_1, \omega_2)\Big|_{\omega_1=0, \omega_2=0}$$

JOINTLY GAUSSIAN RANDOM VARIABLES:

- ✓ Among various standard density function Gaussian density function is the most significantly used density function in the field of science and engineering.
- ✓ In particular it is used to estimate the noise power while calculating the signal to noise ratio.
- ✓ It is some time called bivariate Gaussian density
- ✓ Two random variables are said to be jointly Gaussian if their joint density function of the form

$$f_{XY}(x,y) = \frac{1}{2\pi\, \sigma_X\, \sigma_Y \sqrt{1-\rho^2}}\, exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(x-\bar{X})^2}{\sigma_X{}^2} - \frac{2\,\rho\,(x-\bar{X})\,(y-\bar{Y})}{\sigma_X\,\sigma_Y} + \frac{(y-\bar{Y})^2}{\sigma_Y{}^2}\right]\right\}$$

Here

$$\bar{X} = E[X]$$

$$\bar{Y} = E[Y]$$

$$\sigma_X{}^2 = E(X-\bar{X})^2$$

$$\sigma_Y{}^2 = E(Y-\bar{Y})^2$$

$$\rho = \frac{E[(X-\bar{X})\,(Y-\bar{Y})]}{\sigma_X\,\sigma_Y}$$

1. The maximum value of joint Gaussian density function occurs at $(x = \bar{X}, y = \bar{Y})$

$$max[f_{XY}(x,y)] = \frac{1}{2\pi\,\sigma_X\,\sigma_Y\sqrt{1-\rho^2}}$$

2. If X and Y are two statistical independent random variables then their joint Gaussian density function is

$$f_{XY}(x,y) = \frac{1}{2\pi\,\sigma_X\,\sigma_Y}\ exp\left\{\frac{-1}{2}\left[\frac{(x-\bar{X})^2}{{\sigma_X}^2} + \frac{(y-\bar{Y})^2}{{\sigma_Y}^2}\right]\right\}$$

Observe that if $\rho = 0$, corresponding to uncorrelated X and Y, can be written as

$$f_{XY}(x,y) = f_X(x)\ f_Y(y)$$

Where $f_X(x)\ and\ \ f_Y(y)$ are the marginal density functions of X and Y

$$f_X(x)\ = \frac{1}{\sqrt{2\pi{\sigma_X}^2}}\ exp\left[-\frac{(x-\bar{X})^2}{2{\sigma_X}^2}\right]$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi{\sigma_Y}^2}}\ exp\left[-\frac{(y-\bar{Y})^2}{2{\sigma_Y}^2}\right]$$

Note:

Two random variables are said to be un-correlated if they are statistical independent. However the reverse statement is not true for all cases. But for Gaussian random variables the reverse statement also true.

N Random variables

N random variables $X_1, X_2, \ldots\ldots x_n$ are called jointly Gaussian if their joint density function can be written as

$$f_{X_1\ldots\ldots X_N}(x_{1.}\ldots\ldots x_N) = \frac{|[C_X]^{-1}|^{1/2}}{(2\pi)^{N/2}} exp\left\{-\left[\frac{[x-\bar{X}]^t\,[C_X]^{-1}\,[x-\bar{X}]}{2}\right]\right\}$$

$$[x-\bar{X}] = \begin{bmatrix} x_1-\bar{X}_1 \\ x_2-\bar{X}_2 \\ \vdots \\ x_N-\bar{X}_N \end{bmatrix}$$

$$[C_X] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{2N} & \cdots & C_{NN} \end{bmatrix}$$

$[.]^{-1}$ For the matrix inverse

$[.]^t$ For the matrix transpose

$|[.]|$ For the matrix determinant

Elements of $[\boldsymbol{C_X}]$ called the covariance matrix of N random variables, given by

$$C_{ij} = E\left[(X_i - \bar{X}_i)(X_j - \bar{X}_j)\right] = \begin{cases} {\sigma_{X_i}}^2 & i = j \\ C_{X_iX_j} & i \neq j \end{cases}$$

Properties of Gaussian density function for N random variables:

1. Gaussian random variables are defined by mean, variance and co-variance.
2. All marginal density functions are derived from 'N' variate Gaussian density function are also Gaussian.
3. All conditional density functions derived from 'N' variate Gaussian density function are also Gaussian.
4. Linear transformation Gaussian random variables will also be Gaussian

DESCRPTIVE QUESTIONS

1. Contrast the properties of joint probability distribution function by using necessary mathematical expressions.
2. Contrast the properties of joint probability density function.
3. Explain about marginal distribution functions of random variables X and Y?

4. Explain about marginal density functions of random variables X and Y?
5. Infer with necessary expressions that the density function of sum of two statistically independent random variables is the convolution of individual density functions.
6. Distinguish between various joint moments.
7. Discuss about Joint Central Moments with necessary mathematical Expressions.
8. Interpret the properties of joint characteristic function with the help of necessary mathematical expressions
9. Discuss about the two dimensional Gaussian random variables density function and summarize its properties

1. Two random variables X and Y have the joint PDF

$$f_{XY}(x,y) = Ae^{-(2x+y)} \quad , \quad x, y \geq 0$$
$$0 \quad , \quad otherwise$$

Evaluate (i) A (ii) Marginal pdfs $f_X(x)$ & $f_Y(y)$

2. The joint density of two random variables X and Y is

$$f_{XY}(x,y) = c\,(2x+y), \quad 0 \leq x \leq 1,\ 0 \leq y \leq 2$$
$$0\ , \quad elsewhere$$

Compute
i) The value of c.
ii)The marginal density functions of X and Y.

3. The density function

$$f_{XY}(xy) = \frac{xy}{9}, \quad 0 < x < 2,\ 0 < y < 3$$
$$0\ , \quad elsewhere$$

applies to two random variables X and Y.
Categorize whether X and Y are statistically independent or not.

4. Differentiate whether two given random variables are statistically independent or not if their joint probability density function is given as

$$f_{XY}(x,y) = \frac{5}{16}x^2 y \quad , 0 < x < 2\ \&\ 0 < y < 2$$
$$0 \quad , \quad otherwise$$

5. Two random variables X and Y are having joint density function

$$f_{XY}(x,y) = x + y, \quad 0 < x < 2,\ 0 < y < 1$$
$$0\ , \quad elsewhere$$

Categorize whether X and Y are statistically independent or not.

Calculate correlation coefficient.

6. Given

$$f_{XY}(xy) = \frac{(x+y)^2}{40}, \qquad -1 < x < 1,\ -3 < y < 3$$
$$0, \qquad elsewhere$$

Determine variances of X & Y

7. Two statistically independent random variables X and Y with $\overline{X}=2$, $\overline{X^2}=8$, $\overline{Y}=4$, $\overline{Y^2}=25$. For another random variable given as W = 3X-Y, calculate the variance.

8. X & Y be statistically independent random variables with $\overline{X}=\frac{3}{4}$, $\overline{X^2}=4$, $\overline{Y}=1$, $\overline{Y^2}=5$. If a new random variable is defined as W = X-2Y+1, then calculate
   (i) R_{XY} (ii) R_{XW} (iii) R_{YW} .

9. Two random variables X and Y have means $\bar{X} = 1, \bar{Y} = 3$ variances $\sigma_X^2 = 4$ and $\sigma_Y^2 = 1$ and correlation coefficient ρ_{XY} = 0.4. New random variables W and V are defined such that W = X + 3Y, V = - X +2Y.
   Find (i)Mean (ii)Variance of W and V.

10. Two random variables X and Y have means $\overline{X} = 1$ and $\overline{Y} = 2$ variances $\sigma_X^2 = 4$ and $\sigma_Y^2 = 1$ and a correlation coefficient ρ_{XY} = 0.4. New random variables W and V are defined by V = - X +2Y, W = X + 3Y. Find (i) The means (ii). The Variances (iii) The Correlations (iv) The correlation coefficient ρ_{VW} of V and W.

11. Find & Sketch the density of W = X+Y, if X & Y are Statistically independent and have marginal densities

$$f_X(x) = \frac{1}{a}[u(x) - u(x-a)] \quad f_Y(y) = \frac{1}{b}[u(x) - u(x-b)]$$ assume b>a

1. Calculate 'b' value and also Joint distribution function for the given Joint density function.

$$f_{XY}(x,y) = \begin{cases} b(x+y)^2; & -2 < x \le 2, -3 < y \le 3 \\ 0 & ;\ elsewhere \end{cases}$$

Sol: Given,

$$f_{XY}(x,y) = \begin{cases} b(x+y)^2\ ; & -2 < x \le 2\ , -3 < y \le 3 \\ 0 & ;\quad elsewhere \end{cases}$$

Calculation of "b" value:

We know that,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{f_{XY}(x,y)\,dx\,dy = 1}$$

$$\Rightarrow \int_{-2}^{2}\int_{-3}^{3} b(x+y)^2\,dx\,dy = 1$$

$$\Rightarrow b\int_{-2}^{2}\int_{-3}^{3} (x^2+y^2+2xy)\,dy\,dx = 1$$

Integrating w.r.to 'y'

$$\Rightarrow b\int_{-2}^{2}\left[x^2y+\frac{y^3}{3}+2x\left(\frac{y^2}{2}\right)\right]_{-3}^{3} dx = 1$$

$$\Rightarrow b\int_{-2}^{2}\left[3x^2+\frac{27}{3}+9x-\left(-3x^2-\frac{27}{3}+9x\right)\right] dx = 1$$

$$\Rightarrow b\int_{-2}^{2}[6x^2+18]\,dx = 1$$

$$\Rightarrow 6b\int_{-2}^{2}[x^2+3]\,dx = 1$$

Integrating w.r.to 'x'

$$\Rightarrow 6b\left[\frac{x^3}{3}+3x\right]_{-2}^{2} = 1$$

$$\Rightarrow 6b\left[\frac{8}{3}+6-\left(-\frac{8}{3}-6\right)\right] = 1$$

$$\Rightarrow 6b\left[\frac{16+36}{3}\right]=1$$

$$\Rightarrow 2b[52]=1$$

$$\Rightarrow 104b=1$$

$$\therefore \boldsymbol{b}=\frac{\mathbf{1}}{\mathbf{104}}$$

Calculation of Joint distribution function:

We know that,

$$\boldsymbol{F_{XY}(x,y)}=\int_{-\infty}^{x}\int_{-\infty}^{y}\boldsymbol{f_{XY}(x,y)\,dx\,dy}$$

$$=\int_{-2}^{x}\int_{-3}^{y}b(x+y)^2\,dx\,dy$$

$$=b\int_{-2}^{x}\int_{-3}^{y}(x^2+y^2+2xy)\,dy\,dx$$

Integrating w.r.to 'y'

$$=b\int_{-2}^{x}\left[x^2y+\frac{y^3}{3}+2x\left(\frac{y^2}{2}\right)\right]_{-3}^{y}dx$$

$$=b\int_{-2}^{x}\left[x^2y+\frac{y^3}{3}+xy^2-\left(-3x^2-\frac{27}{3}+9x\right)\right]dx$$

Integrating w.r.to 'x'

$$=b\left[\left(\frac{x^3}{3}\right)y+(x)\frac{y^3}{3}+\left(\frac{x^2}{2}\right)y^2+3\left(\frac{x^3}{3}\right)+9x-9\left(\frac{x^2}{2}\right)\right]_{-2}^{x}$$

$$=b\left[\frac{x^3y}{3}+\frac{xy^3}{3}+\frac{x^2y^2}{2}+x^3+9x-\frac{9x^2}{2}-\left(-\frac{8y}{3}-\frac{2y^3}{3}+2y^2-8-18-18\right)\right]$$

$$\therefore\ \boldsymbol{F_{XY}(x,y) = \frac{1}{104}\left[\frac{(x^3+8)y+(x+2)y^3}{3} + \frac{(y^2-9)x^2}{2} + x^3 + 9x - 2y^2 + 44\right]}$$

2. The Joint density function of two random variables X and Y is given as

$$f_{XY}(x,y) = \begin{cases} x+y\,; & 0 \le x \le 1\,, 0 \le y \le 1 \\ 0 & ; \quad elsewhere \end{cases}$$

Find the Conditional density function.

Sol: Given,

$$f_{XY}(x,y) = \begin{cases} x+y\,; & 0 \le x \le 1\,, 0 \le y \le 1 \\ 0 & ; \quad elsewhere \end{cases}$$

Formulas to find Conditional density function:

$$\boldsymbol{f_X\left(\frac{x}{y}\right) = \frac{f_{XY}(x,y)}{f_Y(y)}}$$

$$\boldsymbol{f_Y\left(\frac{y}{x}\right) = \frac{f_{XY}(x,y)}{f_X(x)}}$$

Calculation of $f_X(x)$ *and* $f_Y(y)$:

We know that,

$$\boldsymbol{f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy}$$

$$\Rightarrow f_X(x) = \int_0^1 (x+y)\, dy$$

$$= \left[xy + \frac{y^2}{2}\right]_0^1$$

$$= x + \frac{1}{2}$$

$$\therefore\ f_X(x) = \frac{2x+1}{2}$$

And

$$\boldsymbol{f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx}$$

$$\Rightarrow f_Y(y) = \int_0^1 (x+y)\,dx$$

$$= \left[\frac{x^2}{2} + xy\right]_0^1$$

$$= y + \frac{1}{2}$$

$$\therefore f_Y(y) = \frac{2y+1}{2}$$

∴ Conditional density functions;

$$\boldsymbol{f_X\left(\frac{x}{y}\right) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{2(x+y)}{2y+1}}$$

$$\boldsymbol{f_Y\left(\frac{y}{x}\right) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{2(x+y)}{2x+1}}$$

3. Two random variables are such that $Y = -4X + 20$ the mean of X is 4. Check whether the given random variables are statistically independent or not, when the variance of X is 2.

Sol: Given,

$$Y = -4X + 20$$

$$E[X] = 4$$

$$\sigma_X{}^2 = 2$$

If two random variables are said to be statistically independent then,

$$\boldsymbol{E[XY] = E[X]E[Y]}$$

Calculation of $E[Y]$:

$$E[Y] = E[-4X + 20]$$

$$= -4E[X] + 20$$

$$= -4(4) + 20$$

$$= -16 + 20$$

$$\therefore E[Y] = 4$$

Evaluation of $E[X^2]$:

We know that,

$$\boldsymbol{\sigma_X}^{\mathbf{2}} = \boldsymbol{m_2} - \boldsymbol{m_1}^{\mathbf{2}}$$

$$\Rightarrow\ 2 = m_2 - 4^2$$

$$\Rightarrow m_2 = 16 + 2$$

$$\therefore\ E[X^2] = 18$$

Calculation of $E[XY]$:

$$E[XY] = E[X(-4X + 20)]$$

$$= E[-4X^2 + 20X]$$

$$= -4E[X^2] + 20E[X]$$

$$= -4(18) + 20(4)$$

$$\therefore E[XY] = 8$$

Now, $E[X]E[Y] = 4 \times 4 = 16$

$$\therefore\ \boldsymbol{E[XY] \neq E[X]E[Y]}$$

Hence, the given random variables X and Y are not statistically independent.

4. Two random variables X and Y have the joint PDF

$$f_{XY}(x,y) = \begin{cases} Ae^{-(2x+y)} & ;\ x,y \geq 0 \\ 0 & ;\ otherwise \end{cases}$$

Evaluate (i) A (ii) Marginal pdfs $f_X(x)$ & $f_Y(y)$

Sol:Given,

$$f_{XY}(x,y) = \begin{cases} Ae^{-(2x+y)} & ;\ x,y \geq 0 \\ 0 & ;\ otherwise \end{cases}$$

(i)*Calculation of* "*A*" *value*:

We know that,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{f_{XY}(x,y)\,dx\,dy = 1}$$

$$\Rightarrow \int_{0}^{\infty}\int_{0}^{\infty} Ae^{-(2x+y)}\,dx\,dy = 1$$

Integrating w.r.to 'y'

$$\Rightarrow A\int_{0}^{\infty}\left[\frac{e^{-(2x+y)}}{-1}\right]_0^{\infty} dx = 1$$

$$\Rightarrow A\int_{0}^{\infty}\left[0-\frac{e^{-2x}}{-1}\right]dx = 1$$

$$\Rightarrow A\int_{0}^{\infty} e^{-2x}\,dx = 1$$

Integrating w.r.to 'x'

$$\Rightarrow A\left[\frac{e^{-2x}}{-2}\right]_0^{\infty} = 1$$

$$\Rightarrow A\left[0-\left(-\frac{1}{2}\right)\right] = 1$$

$$\Rightarrow \frac{A}{2} = 1$$

$$\therefore\ \boldsymbol{A = 2}$$

(ii) *Calculation of Marginal pdfs*:

<u>Formulas</u>:

$$\boldsymbol{f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy}$$

$$\boldsymbol{f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx}$$

Now,

$$f_X(x) = \int_0^\infty Ae^{-(2x+y)}\, dy$$

$$= A\left[\frac{e^{-(2x+y)}}{-1}\right]_0^\infty$$

$$= 2\left[0 - \frac{e^{-2x}}{-1}\right]$$

$$\therefore \boldsymbol{f_X(x) = 2e^{-2x}}$$

Now,

$$f_Y(y) = \int_0^\infty Ae^{-(2x+y)}\, dx$$

$$= A\left[\frac{e^{-(2x+y)}}{-2}\right]_0^\infty$$

$$= 2\left[0 - \frac{e^{-y}}{-2}\right]$$

$$\therefore \boldsymbol{f_Y(y) = e^{-y}}$$

5. The joint density of two random variables X and Y is

$$f_{XY}(x, y) = \begin{cases} c(2x + y) & ;\ 0 \le x \le 1, \quad 0 \le y \le 2 \\ 0 & ;\ elsewhere \end{cases}$$

Compute

(i) The value of "c".

(ii) The marginal density functions of X and Y.

Sol: Given,

$$f_{XY}(x, y) = \begin{cases} c(2x + y) & ;\ 0 \le x \le 1, \quad 0 \le y \le 2 \\ 0 & ;\ elsewhere \end{cases}$$

(i) *Calculation of "c" value*:

We know that,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{f_{XY}(x,y)\,dx\,dy = 1}$$

$$\Rightarrow \int_{0}^{1}\int_{0}^{2} c(2x+y)\,dx\,dy = 1$$

Integrating w.r.to 'y'

$$\Rightarrow c\int_{0}^{1}\left[2xy+\frac{y^2}{2}\right]_0^2 dx = 1$$

$$\Rightarrow c\int_{0}^{1}(4x+2)\,dx = 1$$

Integrating w.r.to 'x'

$$\Rightarrow c\left[\frac{4x^2}{2}+2x\right]_0^1 = 1$$

$$\Rightarrow c[2+2] = 1$$

$$\therefore \boldsymbol{c = \frac{1}{4}}$$

(ii) *Calculation of marginal density functions*:

Formulas:

$$\boldsymbol{f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy}$$

$$\boldsymbol{f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx}$$

Now,

$$f_X(x) = \int_{0}^{2} c(2x+y)dy$$

$$= c\left[2xy + \frac{y^2}{2}\right]_0^2$$

$$= \frac{1}{4}[4x + 2]$$

$$\therefore \boldsymbol{f_X(x) = x + \frac{1}{2}}$$

Now,

$$f_Y(y) = \int_0^1 c(2x + y)\,dx$$

$$= c\left[\frac{2x^2}{2} + xy\right]_0^1$$

$$= \frac{1}{4}[1 + y]$$

$$\therefore \boldsymbol{f_Y(y) = \frac{1}{4}(y + 1)}$$

6. The density function, applies to two random variables X and Y Categorize whether X and Y are statistically independent or not.

$$f_{XY}(x,y) = \begin{cases} \frac{xy}{9} & ; 0 < x < 2, 0 < y < 3 \\ 0 & ; \ elsewhere \end{cases}$$

Sol: Given,

$$f_{XY}(x,y) = \begin{cases} \frac{xy}{9} & ; 0 < x < 2, 0 < y < 3 \\ 0 & ; \ elsewhere \end{cases}$$

<u>Condition for statistical independence</u>: $\boldsymbol{f_{XY}(x,y) = f_X(x) f_Y(y)}$

Calculation of Marginal pdfs:

<u>Formulas</u>:

$$\boldsymbol{f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy}$$

$$\boldsymbol{f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx}$$

Now,

$$f_X(x) = \int_0^3 \frac{xy}{9}\,dy$$

$$= \frac{1}{9}\left[x\left(\frac{y^2}{2}\right)\right]_0^3$$

$$= \frac{1}{9}\left[\frac{9x}{2}\right]$$

$$\therefore\ f_X(x) = \frac{x}{2}$$

Now,

$$f_Y(y) = \int_0^2 \frac{xy}{9}\,dx$$

$$= \frac{1}{9}\left[\left(\frac{x^2}{2}\right)y\right]_0^2$$

$$= \frac{1}{9}[2y]$$

$$\therefore\ f_Y(y) = \frac{2y}{9}$$

$$\therefore\ f_X(x)f_Y(y) = \left(\frac{x}{2}\right)\left(\frac{2y}{9}\right)$$

$$= \frac{xy}{9}$$

$$\therefore\ \boldsymbol{f_X(x)f_Y(y) = f_{XY}(x,y)}$$

Hence, the random variables X and Y are statistically independent.

7. Differentiate whether two given random variables are statistically independent or not, if their joint probability density function is given as

$$f_{XY}(x,y) = \begin{cases} \frac{5}{16}x^2y & ; 0 < x < 2\,, 0 < y < 2 \\ 0 & ; \quad otherwise \end{cases}$$

Sol: Given,

$$f_{XY}(x,y) = \begin{cases} \frac{5}{16}x^2y & ; 0 < x < 2\,, 0 < y < 2 \\ 0 & ; \quad otherwise \end{cases}$$

Condition for statistical independence: $\boldsymbol{f_{XY}(x,y) = f_X(x) f_Y(y)}$

Calculation of Marginal pdfs:

Formulas:

$$\boldsymbol{f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy}$$

$$\boldsymbol{f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx}$$

Now,

$$f_X(x) = \int_0^2 \frac{5}{16}x^2y\, dy$$

$$= \frac{5}{16}\left[x^2\left(\frac{y^2}{2}\right)\right]_0^2$$

$$= \frac{5}{16}[2x^2]$$

$$\therefore\ f_X(x) = \frac{5}{8}x^2$$

Now,

$$f_Y(y) = \int_0^2 \frac{5}{16}x^2y\, dx$$

$$= \frac{5}{16}\left[\left(\frac{x^3}{3}\right)y\right]_0^2$$

$$= \frac{5}{16}\left[\frac{8y}{3}\right]$$

$$\therefore\ f_Y(y) = \frac{5y}{6}$$

$$\therefore\ f_X(x)f_Y(y) = \left(\frac{5}{8}x^2\right)\left(\frac{5y}{6}\right)$$

$$= \frac{25}{48}(x^2y)$$

$$\therefore\ \boldsymbol{f_X(x)f_Y(y) \neq f_{XY}(x,y)}$$

Hence, the given two random variables are not statistically independent.

8. Two random variables X and Y are having joint density function

$$f_{XY}(x,y) = \begin{cases} x+y & ;\ 0<x<2\ , \quad 0<y<1 \\ 0 & ;\ elsewhere \end{cases}$$

Categorize whether X and Y are statistically independent or not.

Sol: Given,

$$f_{XY}(x,y) = \begin{cases} x+y & ;\ 0<x<2\ , \quad 0<y<1 \\ 0 & ;\ elsewhere \end{cases}$$

Condition for statistical independence: $\boldsymbol{f_{XY}(x,y) = f_X(x)f_Y(y)}$

Calculation of Marginal pdfs:

Formulas:

$$\boldsymbol{f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy}$$

$$\boldsymbol{f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx}$$

Now,

$$f_X(x) = \int_0^1 (x+y)\,dy$$

$$= \left[xy + \frac{y^2}{2}\right]_0^1$$

$$\therefore f_X(x) = x + \frac{1}{2}$$

Now,

$$f_Y(y) = \int_0^2 (x+y)\,dx$$

$$= \left[\frac{x^2}{2} + xy\right]_0^2$$

$$\therefore\ f_Y(y) = 2y + 2$$

$$\therefore f_X(x)f_Y(y) = \left(x + \frac{1}{2}\right)(2y+2)$$

$$= 2xy + 2x + y + 1$$

$$\therefore\ \boldsymbol{f_X(x)f_Y(y) \neq f_{XY}(x,y)}$$

Hence, X and Y are not statistically independent.

Calculation of Correlation Coefficient:

Formulas:

$$\boldsymbol{\rho = \left[\frac{\mu_{11}}{\sqrt{\mu_{20}\mu_{02}}}\right] = \left[\frac{C_{XY}}{\sigma_x\sigma_Y}\right]}$$

$$\boldsymbol{\mu_{11}\ or\ C_{XY} = R_{XY} - E[X]E[Y]}$$

$$\boldsymbol{\mu_{20} or\ \sigma_X{}^2 = E[(X-\overline{X})^2] = E[X^2] - (E[X])^2}$$

$$\boldsymbol{\mu_{02}\ or\ \sigma_Y{}^2 = E[(Y-\overline{Y})^2] = E[Y^2] - (E[Y])^2}$$

To find R_{XY}:

$$\boldsymbol{R_{XY} = m_{11} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\,dx\,dy}$$

$$\Rightarrow R_{XY} = \int_0^2 \int_0^1 xy\,(x+y)\,dx\,dy$$

$$\Rightarrow R_{XY} = \int_0^2 \int_0^1 (x^2y + xy^2)\,dx\,dy$$

Integrating w.r.to 'y'

$$\Rightarrow R_{XY} = \int_0^2 \left[x^2\left(\frac{y^2}{2}\right) + x\left(\frac{y^3}{3}\right)\right]_0^1 dx$$

$$\Rightarrow R_{XY} = \int_0^2 \left[\frac{x^2}{2} + \frac{x}{3}\right] dx$$

Integrating w.r.to 'x'

$$\Rightarrow R_{XY} = \left[\frac{x^3}{6} + \frac{x^2}{6}\right]_0^2$$

$$\Rightarrow R_{XY} = \left[\frac{8}{6} + \frac{4}{6}\right]$$

$$\therefore R_{XY} = 2$$

To find $E[X]$:

$$\boldsymbol{E[X] = m_{10} = \int_{-\infty}^{\infty} x f_X(x)\,dx}$$

$$E[X] = \int_0^2 x\left(x + \frac{1}{2}\right) dx$$

$$= \int_0^2 \left(x^2 + \frac{x}{2}\right) dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_0^2$$

$$\therefore\ E[X] = \frac{11}{3}$$

To find$E[Y]$:

$$\boldsymbol{E[Y] = m_{01} = \int_{-\infty}^{\infty} y f_Y(y)\, dy}$$

$$E[Y] = \int_0^1 y(2y+2)\, dy$$

$$= 2\int_0^1 (y^2 + y)\, dy$$

$$= 2\left[\frac{y^3}{3} + \frac{y^2}{2}\right]_0^1$$

$$= 2\left[\frac{5}{6}\right]$$

$$\therefore\ E[Y] = \frac{5}{3}$$

Evaluation ofC_{XY}:

$$\boldsymbol{C_{XY} = R_{XY} - E[X]E[Y]}$$

$$= 2 - \left(\frac{11}{3}\right)\left(\frac{5}{3}\right)$$

$$= 2 - 6.1111$$

$$\therefore \boldsymbol{C_{XY} = -4.1111}$$

To find$E[X^2]$:

$$\boldsymbol{E[X^2] = m_{20} = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx}$$

$$E[X^2] = \int_0^2 x^2\left(x + \frac{1}{2}\right) dx$$

$$= \int_0^2 \left(x^3 + \frac{x^2}{2}\right) dx$$

$$= \left[\frac{x^4}{4} + \frac{x^6}{6}\right]_0^2$$

$$= 4 + \frac{4}{3}$$

$$\therefore\ E[X^2] = \frac{16}{3}$$

To find $E[Y^2]$:

$$\boldsymbol{E[Y^2] = m_{02} = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy}$$

$$E[Y^2] = \int_0^1 y^2(2y+2)\, dy$$

$$= 2\int_0^1 [y^3 + y^2]\, dy$$

$$= 2\left[\frac{y^4}{4} + \frac{y^3}{3}\right]_0^1$$

$$= 2\left[\frac{1}{4} + \frac{1}{3}\right]$$

$$\therefore\ E[Y^2] = \frac{7}{6}$$

Evaluation of $\sigma_X{}^2$:

$$\boldsymbol{\sigma_X{}^2 = E[X^2] - (E[X])^2}$$

$$\sigma_X{}^2 = \frac{16}{3} - \left(\frac{11}{3}\right)^2$$

$$\therefore \boldsymbol{\sigma_X{}^2 = -\frac{73}{9}}$$

Evaluation of $\sigma_Y{}^2$:

$$\boldsymbol{\sigma_Y{}^2 = E[Y^2] - (E[Y])^2}$$

$$\sigma_Y{}^2 = \frac{7}{6} - \left(\frac{5}{3}\right)^2$$

$$\therefore \boldsymbol{\sigma_Y}^{\mathbf{2}} = -\frac{\mathbf{29}}{\mathbf{18}}$$

Correlation coefficient:

$$\boldsymbol{\rho} = \frac{\boldsymbol{C_{XY}}}{\boldsymbol{\sigma_x \sigma_Y}}$$

$$= \frac{-4.1111}{\sqrt{\left(-\frac{73}{9}\right)\left(-\frac{29}{18}\right)}}$$

$$= \frac{-4.1111}{3.615}$$

$$\therefore \ \boldsymbol{\rho} = -\mathbf{1.137}$$

9. Given,

$$f_{XY}(x,y) = \begin{cases} \frac{(x+y)^2}{40} & ; \ -1 < x < 1\,, -3 < y < 3 \\ 0 & ; \ elsewhere \end{cases}$$

Determine variance of$X$ and Y.

Sol: Given,

$$f_{XY}(x,y) = \begin{cases} \frac{(x+y)^2}{40} & ; \ -1 < x < 1\,, -3 < y < 3 \\ 0 & ; \ elsewhere \end{cases}$$

Calculation of Marginal pdfs:

<u>Formulas</u>:

$$\boldsymbol{f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy}$$

$$\boldsymbol{f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx}$$

Now,

$$f_X(x) = \int_{-3}^{3} \frac{(x+y)^2}{40}\,dy$$

$$= \frac{1}{40}\int_{-3}^{3}(x^2+y^2+2xy)\,dy$$

$$= \frac{1}{40}\left[x^2y+\frac{y^3}{3}+2x\left(\frac{y^2}{2}\right)\right]_{-3}^{3}$$

$$= \frac{1}{40}\left[3x^2+\frac{27}{3}+9x-\left(-3x^2-\frac{27}{3}+9x\right)\right]$$

$$= \frac{1}{40}[6x^2+18]$$

$$\therefore\ f_X(x) = \frac{3x^2+9}{20}$$

Now,

$$f_Y(y) = \int_{-1}^{1}\frac{(x+y)^2}{40}\,dx$$

$$= \frac{1}{40}\int_{-1}^{1}(x^2+y^2+2xy)\,dx$$

$$= \frac{1}{40}\left[\frac{x^3}{3}+xy^2+2\left(\frac{x^2}{2}\right)y\right]_{-1}^{1}$$

$$= \frac{1}{40}\left[\frac{1}{3}+y^2+y-\left(-\frac{1}{3}-y^2+y\right)\right]$$

$$= \frac{1}{40}\left[\frac{2}{3}+2y^2\right]$$

$$\therefore\ f_Y(y) = \frac{3y^2+1}{60}$$

To find $m_{10}(or)m_1$ of $f_X(x)$:

$$\boldsymbol{m_{10} = E[X] = m_1 = \int_{-\infty}^{\infty} x\,f_X(x)\,dx}$$

$$m_1 = \int_{-1}^{1} x\left(\frac{3x^2+9}{20}\right)dx$$

$$= \frac{3}{20}\int_{-1}^{1}[x^3 + 3x]\,dx$$

$$= \frac{3}{20}\left[\frac{x^4}{4} + \frac{3x^2}{2}\right]_{-1}^{1}$$

$$= \frac{3}{20}\left[\frac{1}{4} + \frac{3}{2} - \left(\frac{1}{4} + \frac{3}{2}\right)\right]$$

$$= \frac{3}{20}[0]$$

$$\therefore\ m_1 = 0$$

To find $m_{20}(or)m_2$ *of* $f_X(x)$:

$$\boldsymbol{m_{20} = E[X^2] = m_2 = \int_{-\infty}^{\infty} x^2\, f_X(x)\, dx}$$

$$m_2 = \int_{-1}^{1} x^2\left(\frac{3x^2 + 9}{20}\right)dx$$

$$= \frac{3}{20}\int_{-1}^{1}[x^4 + 3x^2]\,dx$$

$$= \frac{3}{20}\left[\frac{x^5}{5} + \frac{3x^3}{3}\right]_{-1}^{1}$$

$$= \frac{3}{20}\left[\frac{1}{5} + 1 - \left(-\frac{1}{5} - 1\right)\right]$$

$$= \frac{3}{20}\left[\frac{12}{5}\right]$$

$$\therefore m_2 = 0.36$$

Variance of X:

$$\boldsymbol{\sigma_X{}^2 = m_2 - m_1{}^2}$$

$$\sigma_X{}^2 = 0.36 - 0^2$$

$$\therefore \boldsymbol{\sigma_X{}^2 = 0.36}$$

To find $m_{01}(or)m_1$ *of* $f_Y(y)$:

$$\boldsymbol{m_{01} = E[Y] = m_1 = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy}$$

$$m_1 = \int_{-3}^{3} y \left(\frac{3y^2 + 1}{60}\right) dy$$

$$= \frac{1}{60} \int_{-3}^{3} (3y^3 + y)\, dy$$

$$= \frac{1}{60} \left[3\left(\frac{y^4}{4}\right) + \frac{y^2}{2}\right]_{-3}^{3}$$

$$= \frac{1}{60} \left[\frac{3}{4}(81) + \frac{9}{2} - \left(\frac{3}{4}(81) + \frac{9}{2}\right)\right]$$

$$= \frac{1}{60}[0]$$

$$\therefore\ m_1 = 0$$

To find $m_{02}(or)m_2$ *of* $f_Y(y)$:

$$\boldsymbol{m_{02} = E[Y^2] = m_2 = \int_{-\infty}^{\infty} y^2\, f_Y(y)\, dy}$$

$$m_2 = \int_{-3}^{3} y^2 \left(\frac{3y^2 + 1}{60}\right) dy$$

$$= \frac{1}{60} \int_{-3}^{3} (3y^4 + y^2)\, dy$$

$$= \frac{1}{60} \left[3\left(\frac{y^5}{5}\right) + \frac{y^3}{3}\right]_{-3}^{3}$$

$$= \frac{1}{60} \left[\frac{3}{5}(243) + \frac{27}{3} - \left(\frac{3}{5}(-243) - \frac{27}{3}\right)\right]$$

$$= \frac{1}{60}[309.6]$$

$$\therefore\ m_2 = 5.16$$

Variance of Y:

$$\boldsymbol{\sigma_Y{}^2 = m_2 - m_1{}^2}$$

$$\sigma_Y{}^2 = 5.16 - 0^2$$

$$\therefore\ \boldsymbol{\sigma_Y{}^2 = 5.16}$$

10. Two statistically independent random variables X and Y have $\bar{X} = 2, \overline{X^2} = 8, \bar{Y} = 4, \overline{Y^2} = 25.$

For another random variable given as $W = 3X - Y$, calculate the variance.

Sol:Given,

$$W = 3X - Y \text{ and}$$

$$\bar{X} = 2,\ \ \overline{X^2} = 8,\ \ \bar{Y} = 4,\ \ \overline{Y^2} = 25$$

To find m_1 *of* W:

$$m_1 = E[W] = E[3X - Y]$$

$$= 3E[X] - E[Y]$$

$$= 3(2) - 4$$

$$\therefore\ m_1 = 2$$

To find m_2 *of* W:

$$m_2 = E[W^2] = E[(3X - Y)^2]$$

$$= E[9X^2 + Y^2 - 6XY]$$

$$= 9E[X^2] + E[Y^2] - 6E[XY]$$

$$= 9(8) + 25 - 6E[X]E[Y][\because X \text{ and } Y \text{ are statistically independent }]$$

$$= 72 + 25 - 6(2)(4)$$

$$\therefore m_2 = 49$$

Variance of W:

$$\boldsymbol{\sigma_W{}^2 = m_2 - m_1{}^2}$$

$$\sigma_W{}^2 = \boldsymbol{49 - 2^2}$$

$$\therefore\ \boldsymbol{\sigma_W{}^2 = 45}$$

11. X & Y be statistically independent random variables with $\bar{X} = \frac{3}{4}$, $\overline{X^2} = 4$, $\bar{Y} = 1$, $\overline{Y^2} = 5$. If a new random variable is defined as $W = X - 2Y + 1$ then, calculate

(i)R_{XY} (ii)R_{XW} (iii)R_{YW}

Sol:Given,

$W = X - 2Y + 1$ and

$\bar{X} = \frac{3}{4}, \overline{X^2} = 4, \bar{Y} = 1, \overline{Y^2} = 5$

To find mean of W:

$$E[W] = E[X - 2Y + 1]$$

$$= E[X] - 2E[Y] + 1$$

$$= \frac{3}{4} - 2(1) + 1$$

$$\therefore \; E[W] = -\frac{1}{4}$$

(i)R_{XY}:

$$\boldsymbol{R_{XY} = E[XY]}$$

$$= \boldsymbol{E[X]E[Y]}[\because X \; and \; Y \; are \; statistically \; independent \;]$$

$$\Rightarrow R_{XY} = \bar{X}\bar{Y}$$

$$\Rightarrow R_{XY} = \frac{3}{4} \times 1$$

$$\therefore \; \boldsymbol{R_{XY} = \frac{3}{4}}$$

(ii)R_{XW}:

$$R_{XW} = E[XW]$$

$$= E[X(X - 2Y + 1)]$$

$$= E[X^2 - 2XY + X]$$

$$= E[X^2] - 2E[X]E[Y] + E[X]$$

$$= 4 - 2\left(\frac{3}{4}\right)(1) + \frac{3}{4} = \frac{13}{4}$$

$\therefore\ \boldsymbol{R_{XW} = 3.25}$

(iii)R_{YW}:

$$R_{YW} = E[YW]$$
$$= E[Y(X - 2Y + 1)]$$
$$= E[YX - 2Y^2 + Y]$$
$$= E[Y]E[X] - 2E[Y^2] + E[Y]$$
$$= (1)\left(\frac{3}{4}\right) - 2(5) + 1$$
$$= \frac{-33}{4}$$
$$\therefore\ \boldsymbol{R_{YW} = -8 \cdot 25}$$

12. Two random variables X and Y have means $\bar{X} = 1, \bar{Y} = 3$ and variances ${\sigma_X}^2 = 4$ and ${\sigma_Y}^2 = 1$ and correlation coefficient ρ_{XY}=0.4. New random variables W and V are defined such that $W = X + 3Y$ and $V = -X + 2Y$.

    Find (i)Mean (ii)Variance of W and V

Sol:Given,

$\bar{X} = 1, \bar{Y} = 3$

$${\sigma_X}^2 = 4,\ {\sigma_Y}^2 = 1$$

$\rho_{XY} = 0.4$ and also

$$W = X + 3Y, V = -X + 2Y$$

(i) *<u>Mean of W</u>*:

$$E[W] = \bar{W} = E[X + 3Y]$$
$$= E[X] + 3E[Y]$$
$$= \bar{X} + 3\bar{Y}$$
$$= 1 + 3(3)$$
$$\therefore \boldsymbol{E[W] = 10}$$

<u>Mean of V</u>:

$$E[V] = \bar{V} = E[-X + 2Y]$$

$$= -\ E[X] + 2E[Y]$$

$$= -\bar{X} + 2\bar{Y}$$

$$= -1 + 2(3)$$

$$\therefore \boldsymbol{E[V]\ = 5}$$

(ii)<u>*Variance of* W *and* V</u>:

Given,

$$\rho_{XY} = 0.4 \left[\because \boldsymbol{\rho_{XY} = \frac{C_{XY}}{\sigma_X \sigma_Y}}\right]$$

$$\Rightarrow \frac{C_{XY}}{\sigma_X \sigma_Y} = 0.4$$

$$\Rightarrow \frac{R_{XY} - E[X]E[Y]}{\sigma_X \sigma_Y} = 0.4$$

$$\Rightarrow E[XY] = 0.4(\sigma_X \sigma_Y) + E[X]E[Y]$$

$$\Rightarrow R_{XY} =\ E[XY] = 0.4(2 \times 1) + 3$$

$$\therefore\ R_{XY} =\ E[XY] = 3.8$$

Now,

$$\sigma_X{}^2 = m_2 - m_1^2$$

$$\Rightarrow m_2 =\ \sigma_X{}^2 + m_1^2$$

$$\Rightarrow m_2 = 4 + 1$$

$$\therefore\ E[X^2] = 5$$

And

$$\sigma_Y{}^2 = m_2 - m_1^2$$

$$\Rightarrow m_2 =\ \sigma_Y{}^2 + m_1^2$$

$$\Rightarrow m_2 = 1 + 9$$

$$\therefore\ E[Y^2] = 10$$

To find variance of W:

Now, m_2 of W i.e,

$$E[W^2] = E[(X + 3Y)^2]$$

$$= E[X^2 + 9Y^2 + 6XY]$$

$$= E[X^2] + 9\,E[Y^2] + 6E[XY]$$

$$= 5 + 90 + 6(3.8)$$

$$\therefore\ E[W^2] = 117.8$$

Variance of W:

$$\sigma_W{}^2 = m_2 - m_1^2$$

$$= E[W^2] - E[W]^2$$

$$= 117.8 - 10^2$$

$$\therefore\ \boldsymbol{\sigma_W{}^2 = 17.8}$$

To find variance of V:

Now, m_2 of V i.e,

$$E[V^2] = E[(-X + 2Y)^2]$$

$$= E[X^2 + 4Y^2 - 4XY]$$

$$= E[X^2] + 4\,E[Y^2] - 4E[XY]$$

$$= 5 + 40 - 4(3.8)$$

$$\therefore\ E[V^2] = 29.8$$

Variance of V:

$$\sigma_V{}^2 = m_2 - m_1^2$$

$$= E[V^2] - E[V]^2$$

$$= 29.8 - 5^2$$

$$\therefore\ \boldsymbol{\sigma_V{}^2 = 4.8}$$